ON THE DRAWING OF A PLASTIC STRIP

(O VOLOCHENII PLASTICHESKOI POLOSY)

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The present paper considers the problem of the drawing of a plastic strip through the rigid walls of a die. The solution of this problem is based on a plasticity condition of a special type [1], and is in simple closed form.

As usual, plane-plastic equilibrium is defined by the differential equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \tag{1}$$

and a plasticity condition of a special form

$$\tfrac{1}{4}(\sigma_x - \sigma_y)^2 + \tau_{xy}^2 = k^2 \sin^2\left(\frac{\sigma_x + \sigma_y}{2k} + \frac{m}{k}\right) \tag{2}$$

with

$$\frac{\pi k}{2} \leqslant \tfrac{1}{2}(\sigma_x + \sigma_y) + m \leqslant \pi k$$

It is convenient to transform this system of equations from the components σ_x, σ_y and τ_{xy} to new variables ψ and ϕ by means of the formulas

$$\left.\begin{matrix}\sigma_x \\ \sigma_y\end{matrix}\right\} = k(2\psi \pm \sin 2\psi \cos 2\varphi) - m, \qquad \tau_{xy} = k \sin 2\psi \sin 2\varphi \tag{3}$$

Introducing Expressions (3) into the differential equations (1), we obtain a system of two equations of the first order for the two new variables ψ and ϕ. It is of hyperbolic type, and after the introduction of the variables

$$\xi = \psi + \varphi, \qquad \eta = \psi - \varphi \quad \text{or} \quad 2\psi = \xi + \eta, \qquad 2\varphi = \xi - \eta \tag{4}$$

it can be transformed to two differential equations of the following form:

$$\cos\eta\frac{\partial\eta}{\partial x}-\sin\eta\frac{\partial\eta}{\partial y}=0,\qquad \cos\xi\frac{\partial\xi}{\partial x}+\sin\xi\frac{\partial\xi}{\partial y}=0 \tag{5}$$

The equations of the characteristics

$$x\sin\eta+y\cos\eta=\text{const},\qquad \eta=\text{const}$$
$$x\sin\xi-y\cos\xi=\text{const},\qquad \xi=\text{const}$$

determine two families of straight lines in the xy-plane. These lines enclose the angles $\phi\mp\psi$ with the x-axis and intersect at the variable angles 2ψ. The basic system of equations (5) has the integrals

$$x\sin\eta+y\cos\eta=f(\eta),\qquad x\sin\xi-y\cos\xi=g(\xi) \tag{6}$$

which contain the arbitrary functions $f(\eta)$ and $g(\xi)$.

In addition, there exist simple particular integrals corresponding to constant ξ and η.

At the outset we take note of the particular integrals

$$\xi=\xi_0,\qquad x\sin\eta+y\cos\eta=f(\eta) \tag{7}$$

which have a net of characteristics consisting of nonparallel straight lines η = const and a family of parallel straight lines.

Further, we note the particular integrals

$$\eta=\eta_0,\qquad x\sin\xi-y\cos\xi=g(\xi) \tag{8}$$

which have a net of characteristics formed by a family of parallel straight lines and nonparallel straight lines ξ = const.

Finally, we note the particular integrals $\xi=\xi_0$, $\eta=\eta_0$, which have a net of characteristics consisting of two families of parallel straight lines.

It is not difficult to see that along the characteristics η = const, inclined at the angle $\phi-\psi$ to the x-axis, the normal stresses σ_n and the shear stresses τ have the form

$$\sigma_n=k(2\psi-\sin 2\psi\cos 2\psi)-m,\qquad \tau=k\sin^2 2\psi$$

In an analogous way, it is easy to show that along the characteristics ξ = const, which form the angles $\phi+\psi$ with the x-axis, the normal stresses σ_n and the shear stresses τ will be

$$\sigma_n = k\,(2\psi - \sin 2\psi \cos 2\psi) - m, \qquad \tau = -k \sin^2 2\psi$$

We now investigate the distribution of stresses in a plastic strip which is drawn through the smooth rigid walls of a die [1,2], assuming that the walls form an angle of 2γ, and that the drawing force and back-pull are $2P$ and $2Q$, respectively. Because of symmetry with respect to the x-axis it is necessary to study only the upper half of the strip, bearing in mind that along the x-axis $\tau_{xy} = 0$ and $\phi = 0$.

First, we investigate the stress field in the plastic strip, which consists of triangular and quadrilateral regions, shown in Fig. 1; further we assume that there acts a uniformly distributed normal pressure all along the contact segment AB.

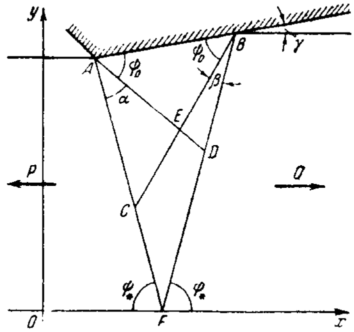

Fig. 1.

The coordinates of the point A we denote by x_1, y_1 and those of point B by x_2, y_2.

It is not difficult to determine the quantities ξ and η in the regions AEB, CAE, EBD and $CEDF$ with the help of the particular integrals (7) and (8). The quantities ξ and η must satisfy Equations (5) and be continuous on the lines AED and BEC. The relative contraction of the thickness, that is, the compression of the strip r, is determined by the following formula:

$$r = 1 - \frac{y_1}{y_2} = 1 - \frac{\sin(\psi_* - \gamma)}{\sin(\psi_* + \gamma)}$$

and the angles α and β will be

$$\alpha = \psi_* - \psi_0 + \gamma, \qquad \beta = \psi_* - \psi_0 - \gamma$$

In the region AEB $\psi = \psi_0$, $\phi = \gamma$, the quantities ξ and η are constants

$$\xi = \psi_0 + \gamma, \qquad \eta = \psi_0 - \gamma$$

and the net of characteristics is formed of two families of parallel straight lines inclined to the x-axis at the angles $\psi_0 \mp \gamma$.

In the region CAE the quantities ξ and η are determined by

$$\xi = \psi_0 + \gamma, \qquad \tan \eta = -\frac{y - y_1}{x - x_1}$$

and the net of characteristics consists of a bundle of straight lines passing through the point A and straight lines parallel to BC.

In the region EBD the quantities ξ and η are expressed as follows:

$$\tan \xi = \frac{y - y_2}{x - x_2}, \qquad \eta = \psi_0 - \gamma$$

and the net of characteristics consists of a bundle of straight lines passing through the point B and straight lines parallel to AD.

In the region $CEDF$ the quantities ξ and η are

$$\tan \xi = \frac{y - y_2}{x - x_2}, \qquad \tan \eta = -\frac{y - y_1}{x - x_1}$$

and the net of characteristics is formed by two bundles of straight lines passing through the points A and B.

The horizontal component of the resultant of all forces acting on the section ACF must be equal to the drawing force P. Hence it follows that

$$P = \int_0^{y_1} [\sigma_n + \tau \cot \psi_*]\, dy, \qquad P = py_1$$

or

$$(p + m)\, y_1 = \int_0^{y_1} [(\sigma_n + m) + \tau \cot \psi_*]\, dy$$

It is clear that along the entire segment AF the quantity $\eta = \psi$, and the stress components σ_n and τ can be expressed as

$$\sigma_n + m = k\, [(\xi + \psi_*) - \sin (\xi + \psi_*) \cos (\xi + \psi_*)], \qquad \tau = k \sin^2 (\xi + \psi_*)$$

Further, it is not difficult to show that the quantity $\xi = \psi_0 + \gamma$ along the segment AC, the quantity $\xi = \psi_*$ at the point F, and along CF there exists the relationship

$$y - y_1 = -y_1 \sin 2\psi_* \frac{\sin (\xi - \gamma)}{\sin (\psi_* - \gamma) \sin (\xi + \psi_*)}$$

Carrying out the preceding integration, we obtain after certain transformations

$$p+m=2k\left\{\psi_*+\frac{\cos\psi_*}{\sin(\psi_*-\gamma)}\left[\sin\psi_0\sin(\psi_0+\gamma)-(\psi_0+\gamma-\psi_*)\sin\gamma\right]\right\} \quad (9)$$

and for $\psi_* = \pi/2$ we have $p + m = \pi k$.

The horizontal component of the resultant of all forces applied over the cross-section BDF must be equal to the back-pull Q. Hence it follows that

$$Q=\int_0^{y_2}[\sigma_n-\tau\cot\psi_*]\,dy, \qquad Q=qy_2$$

or

$$(q+m)\,y_2=\int_0^{y_2}[(\sigma_n+m)-\tau\cot\psi_*]\,dy$$

It is obvious that along the entire segment BF the quantity $\xi = \psi_*$ and the stress components are

$$\sigma_n+m=k\left[(\psi_*+\eta)-\sin(\psi_*+\eta)\cos(\psi_*+\eta)\right], \qquad \tau=-k\sin^2(\psi_*+\eta)$$

Moreover, it is easy to show that the quantity $\eta = \psi_0 - \gamma$ on the segment BD, that, as before, the quantity $\eta = \psi_*$ at the point F, and that along DF there exists the relationship

$$y-y_2=-y_2\sin 2\psi_*\frac{\sin(\gamma+\eta)}{\sin(\psi_*+\gamma)\sin(\psi_*+\eta)}$$

Carrying out the integration we find, analogously to the previous case

$$q+m=2k\left\{\psi_*+\frac{\cos\psi_*}{\sin(\psi_*+\gamma)}\left[\sin\psi_0\sin(\psi_0-\gamma)+(\psi_0-\gamma-\psi_*)\sin\gamma\right]\right\} \quad (10)$$

and for $\psi_* = \pi/2$ we have $q + m = \pi k$.

The horizontal component of the resultant pressure along the contact segment AB must equilibrate the difference $P - Q$. Hence there follows the equation

$$Q-P=\sigma_n(y_2-y_1)$$

or

$$(q+m)\,y_2-(p+m)\,y_1=(\sigma_n+m)(y_2-y_1)$$

It is obvious that along the segment AB there is applied the normal stress

$$\sigma_n + m = k(2\psi_0 - \sin 2\psi_0)$$

Therefore, after simple transformations we obtain

$$(q-p)\tan\psi_* + (p+q+2m)\tan\gamma = 2k(2\psi_0 - \sin 2\psi_0)\tan\gamma$$

It is easy to see that this equation is a consequence of Equations (9) and (10), which were introduced earlier.

We remark that the solution which has been constructed is valid as long as the angles

$$\beta \geqslant 0, \qquad \psi_0 \geqslant \tfrac{1}{4}\pi, \qquad \psi_* \geqslant \tfrac{1}{4}\pi$$

and as long the following condition is fulfilled:

$$\psi_* - \psi_0 \geqslant \gamma$$

The particular case when the side BEC coincides with BDF, and the region $CEDF$ degenerates into the section DF, corresponds to

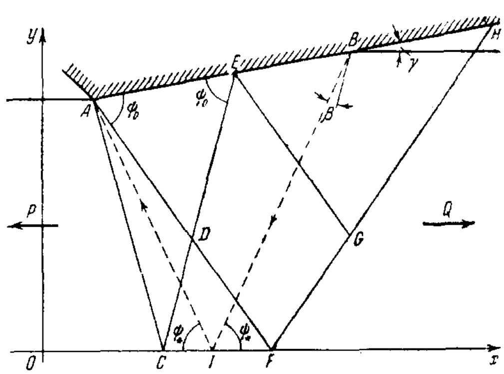

Fig. 2.

$$\beta = 0, \qquad \psi_* - \psi_0 = \gamma, \qquad p + m = k(2\psi_* + \sin 2\psi_*)$$

We examine, next, another stress field in a plastic strip consisting of triangular and quadrilateral regions as indicated in Fig. 2. We shall assume that a uniformly distributed normal pressure acts on the contact segment AE.

Here, it is easy to find the quantities ξ and η in the regions ADE, CAD, CDF, $DEFG$ and EGH with the aid of the general integrals (6) and the particular integrals (7) and (8). These quantities, ξ and η, should satisfy Equations (5) and be continuous along the lines ADF, CDE and EG.

The relative contraction of the strip thickness, that is, the compression r as before, is expressed as

$$r = 1 - \frac{y_1}{y_2} = 1 - \frac{\sin(\psi_* - \gamma)}{\sin(\psi_* + \gamma)}$$

and the angle as

$$\beta = \psi_0 - \psi_* + \gamma$$

In the region ADE the angles $\psi = \psi_0$, $\phi = \gamma$, and the quantities ξ and η are constants

$$\xi = \psi_0 + \gamma, \qquad \eta = \psi_0 - \gamma$$

and the net of characteristics consists of two families of parallel straight lines inclined at the angles $\psi_0 \mp \gamma$ to the x-axis.

In the region CAD the quantities ξ and η are determined in the following manner:

$$\xi = \psi_0 + \gamma, \qquad \tan\eta = -\frac{y - y_1}{x - x_1}$$

and the net of characteristics consists of a bundle of straight lines passing through the point A and straight lines parallel to CD.

In the region CDF the quantities ξ and η are expressed as

$$\tan\xi = \frac{y + y_1}{x - x_1}, \qquad \tan\eta = -\frac{y - y_1}{x - x_1}$$

and the net of characteristics is formed by two bundles of straight lines passing through the point A and the point symmetrical to A.

The horizontal component of the resultant of the forces applied across the section AC must be equal to the drawing force P. Hence it follows that

$$p + m = k\,[2(\psi_0 + \gamma) + \sin 2(\psi_0 + \gamma)] \tag{11}$$

In the region $DEGF$ the quantities ξ and η are given by

$$\tan\xi = \frac{y + y_1}{x - x_1}, \qquad \eta = \psi_0 - \gamma$$

and the net of characteristics is formed by a bundle of straight lines through the point symmetrical to A and straight lines parallel to DF.

In the region EGH the quantities ξ and η are

$$\tan\xi = \frac{y + y_1}{x - x_1}, \qquad \eta = -\frac{y - y_1 - 2y_1\cos 2\gamma}{x - x_1 + 2y_1\sin 2\gamma}$$

and the net of characteristics consists of two families of nonparallel straight lines.

It should be noted that the solution which has been constructed has physical meaning only in those portions of the regions CDF, $DEGF$ and EGH which lie to the left of the straight line IB.

The horizontal component of the resultant of the pressure along the contact section AB must equilibrate the difference $P - Q$. Hence there follows the equation

$$Q - P = \int_{y_1}^{y_2} \sigma_n dy$$

or

$$(q + m) y_2 - (p + m) y_1 = \int_{y_1}^{y_2} (\sigma_n + m) dy$$

It is evident that along the section AE there is applied the normal stress

$$\sigma_n + m = k(2\psi_0 - \sin 2\psi_0)$$

while along the section EB there is applied the normal stress

$$\sigma_n + m = k(2\psi - \sin 2\psi)$$

where the angle ψ is related to y by

$$y - y_1 = 2y_1 \frac{\sin\gamma\cos(\psi+\gamma)}{\sin\psi}$$

Calculating the foregoing integral, we have, after some simple transformations

$$(q + m)\sin(\psi_* + \gamma) - (p + m)\sin(\psi_* - \gamma) = \tag{12}$$
$$= 4k\sin\gamma\{(\psi_* - \gamma)\cos(\psi_* - \gamma)\cos\gamma - \sin(\psi_* - \gamma)[\psi_0\sin\gamma + \cos\psi_0\cos(\psi_0 + \gamma)]\}$$

We note that the solution which has been constructed is valid as long as the angles

$$0 \leqslant \beta \leqslant 2\gamma, \quad \psi_0 \geqslant \frac{1}{4}\pi, \quad \psi_* \geqslant \frac{1}{4}\pi$$

and as long as the following condition is met:

$$-\gamma \leqslant \psi_* - \psi_0 \leqslant \gamma$$

The particular case when the side IB coincides with the side CE

corresponds to

$$\beta = 0, \qquad \psi_* - \psi_0 = \gamma,$$
$$p + m = k(2\psi_* + \sin 2\psi_*)$$

The results of computations for the angle $\gamma = 10°$ and for various values of the angle ψ_0 from 45° to 80° in 5° increments are indicated below.

The values $\overline{p} = (p + m)/k$ and $\overline{q} = (q + m)/k$ as functions of r, determined by Formulas (9) and (10) or by Formulas (11) and (12), are indicated in Figs. 3 and 4 by solid lines. Values of $\overline{p}$ and $\overline{q}$ corresponding to $\psi_* - \psi_0 = \gamma$ and $\psi_0 - \psi_* = \gamma$ are connected by the dotted curves I and II.

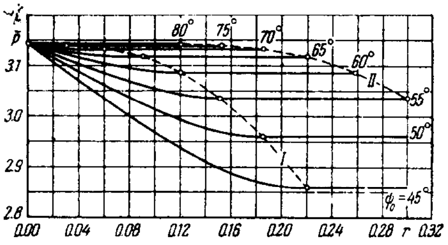

Fig. 3.

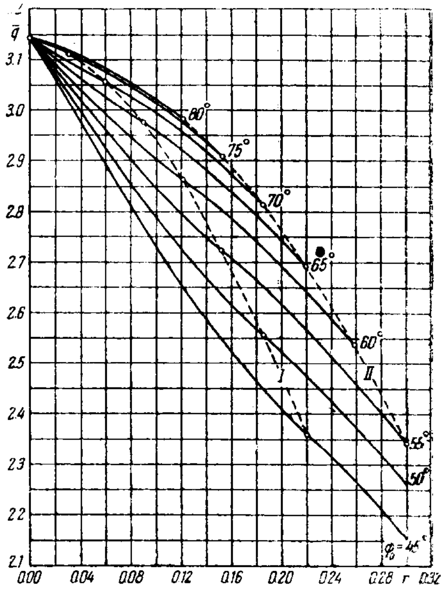

Fig. 4.

In conclusion, we remark that, along with the stress field, it is not difficult to construct the velocity field in a plastic strip. Moreover, it is easy to show that there are no inconsistencies between these fields, and that they completely agree with one another.

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